Proceedings of the Pakistan Academy of Sciences 52 (4): 381–388 (2015)

ISSN: 0377 - 2969 (print), 2306 - 1448 (online)

Pakistan Academy of Sciences

Research Article

An Efficient Algorithm for Nonlinear Fractional Partial Differential Equations

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Abstract: In this paper, we used a newly developed fractional complex transform to convert the given fractional partial differential equations into corresponding partial differential equations and subsequently Variational Iteration Method is applied on the transformed partial differential equations. Three examples are illustrated to show the effectiveness and applicability of the transform. The obtained results are very audacious and hence the same can be applied to other mathematical problems.

Keywords: Fractional differential equation; Fractional complex transform; Variational iteration method
AMS Mathematics Subject Classification: 35R11.

1. INTRODUCTION

Fractional calculus of arbitrary order [1-2] has been used to model much physical and engineering process that are found to be best described by fractional differential equations. Considerable attention has been given to the solution of fractional differential equations, integral equations and system of fractional partial differential equations of physical interest. Most fractional differential equations do not have exact analytic solutions, analytic and numerical techniques, therefore, are used extensively for the solutions of such problems. The detailed study of literature reflects the implementation of wide range of numerical and analytical techniques (including Finite Difference [3-5], Adomian's Decomposition [6-10], Exp function [11-12], Homotopy Perturbation [13], Reduce Differential Transform [14], Homotopy Analysis [15], and Variational Iteration [16-17] for the solutions of linear and nonlinear equations of fractional-order. Inspired and motivated by the ongoing research in this area, we use a fractional complex transform (FCT) [18-19] in order to convert the given fractional partial differential equations (FPDEs) into corresponding partial differential equations (PDEs); subsequently Variational Iteration Method (VIM) is applied on the transformed PDEs and inverse transformation yields the results it in terms of original variables. Computational work re-confirms that the proposed algorithm is highly efficient, fully compatible, and extremely appropriate for factional PDEs arising in mathematical physics and hence can be extended to other problems of diversified nonlinear nature. In particular, we focus our attention on three very important equations which are called Benjamin–Bona–Mahoney (BM) Equation [20], Cahn-Hilliard equation [21] and Gardner equation [22]. Numerical results are very encouraging.

2. FRACTIONAL COMPLEX TRANSFORM

The fractional complex transform was first proposed [23] and is defined as

$$\begin{cases} T = \dfrac{p\,t^{\alpha}}{\Gamma(\alpha+1)} \\ X = \dfrac{q\,x^{\beta}}{\Gamma(\beta+1)} \\ Y = \dfrac{k\,y^{\gamma}}{\Gamma(1+\gamma)} \\ Z = \dfrac{l\,z^{\delta}}{\Gamma(1+\delta)} \end{cases} \tag{1}$$

where *p*, *q*, *k*, and *l* are unknown constants, $0<\alpha\le 1,\quad 0<\beta\le 1,\quad 0<\gamma\le 1,\quad 0<\delta\le 1.$

Received, December 2014; Accepted, September 2015
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3. VARIATIONAL ITERATION METHOD

To illustrate the basic concept of variational iteration method, we consider the following general nonlinear differential equation given in the form:

$$Lu(t)+Nu(t)=g(t), \tag{2}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a known analytical function. We can construct a correction functional as

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(Lu_n(\xi)+N\tilde{u}_n(\xi)-g(\xi))d\xi \tag{3}$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory, the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation, namely $\delta\tilde{u}_n=0$. It is obvious that the successive approximation $u_j,\ j\geq 0$ can be established by determining general Lagrange's multiplier λ, which can be identified optimally via the variational theory. Therefore, we first determine Lagrange's multiplier that will be identified optimally via integration by parts. The successive approximation of the $u_{n+1}(x,t),\ n\geq 0$ solution $u(x,t)$ will be readily obtained upon using the Lagrange's multiplier and by using any selective function u_0. The initial values are usually used for selecting the zeroth approximation u_0. With λ determined, several approximations $u_j,\ j\geq 0$ follows immediately. Consequently, the exact solution may be obtained by using

$$u(x,t)=\lim_{x\to\infty} u_n(x,t). \tag{4}$$

4. SOLUTION PROCEDURE

4.1 Benjamin–Bona–Mahoney (BM) Equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^3 u}{\partial^2 x\partial t}+\frac{\partial u}{\partial x}+u\left(\frac{\partial u}{\partial x}\right)=0, \tag{5}$$

with the initial condition $u(x,0)=\sec h^2\left(\frac{x}{4}\right).$

Applying the transformation [23], we get the following partial differential equation

$$\frac{\partial u}{\partial T}-\frac{\partial^3 u}{\partial^2 x\partial t}+\frac{\partial u}{\partial x}+u\left(\frac{\partial u}{\partial x}\right)=0, \tag{6}$$

The correction functional can be written in the form:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^T \lambda(s)\left(\frac{\partial u}{\partial T}-\frac{\partial^3 u}{\partial^2 x\partial t}+\frac{\partial u}{\partial x}+u\left(\frac{\partial u}{\partial x}\right)\right)ds. \tag{7}$$

The stationary conditions yield

$1+\lambda=0,\quad \lambda'=0.$

This in turn gives

$$\lambda=-1. \tag{8}$$

Hence (7) takes the form

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^T\left(\frac{\partial u}{\partial T}-\frac{\partial^3 u}{\partial^2 x\partial t}+\frac{\partial u}{\partial x}+u\left(\frac{\partial u}{\partial x}\right)\right)ds. \tag{9}$$

Consequently,

$$u_0(x,T)=\sec h^2\left(\frac{x}{4}\right),\;\; u_1(x,T)=\sec h^2\left(\frac{x}{4}\right)+\frac{1}{2}\sec h^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)T-\frac{1}{4}\sec h^4\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)T,$$

$$u_2(x,T)=\sec h^2\left(\frac{x}{4}\right)+\frac{1}{2}\sec h^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)T$$

$$-\frac{1}{2}\begin{pmatrix}\frac{1}{2}\sec h^2\left(\frac{x}{4}\right)\left(\frac{1}{4}-\frac{1}{4}\tanh^2\left(\frac{x}{4}\right)\right)+\frac{1}{4}\sec h^4\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)\\ -\frac{1}{2}\sec h^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)\left(\frac{1}{4}-\frac{1}{4}\tanh^2\left(\frac{x}{4}\right)\right)-\frac{1}{4}\sec h^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)\end{pmatrix}T^2+\frac{1}{2}\sec h^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right),$$

$\vdots$

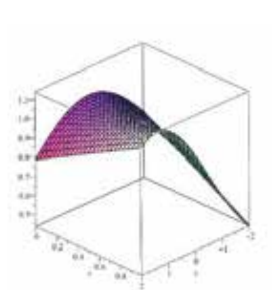
The inverse transformation will yield

$$u_2(x,t)=\sec h^2\left(\frac{x}{4}\right)+\frac{1}{2}\sec h^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

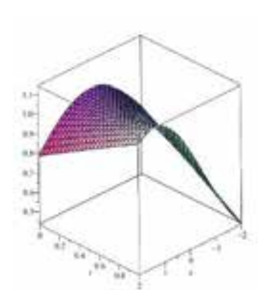
$$-\frac{1}{2}\begin{pmatrix}\frac{1}{2}\sec h^2\left(\frac{x}{4}\right)\left(\frac{1}{4}-\frac{1}{4}\tanh^2\left(\frac{x}{4}\right)\right)+\frac{1}{4}\sec h^4\left(\frac{x}{4}\right).\\ \tanh^3\left(\frac{x}{4}\right)-\frac{1}{2}\sec h^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)\left(\frac{1}{4}-\frac{1}{4}\tanh^2\left(\frac{x}{4}\right)\right)\\ -\frac{1}{4}\sec h^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)\end{pmatrix}\frac{t^{2\alpha}}{\Gamma^2(\alpha+1)} \tag{10}$$

$$+\frac{1}{2}\sec h^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

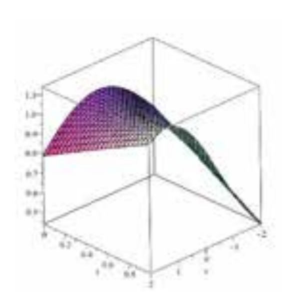
Graphical representation of approximate and exact solutions of (5) for different values of α, using only three iterations of the VIM solution



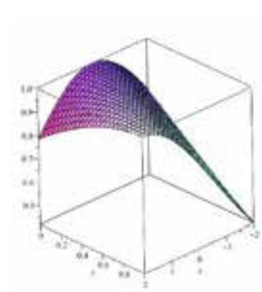
$(a)\,\alpha=0.4$



$(b)\,\alpha=0.8$



$(c)\,\alpha=1$



(d) *Exact solution*

4.2 Cahn-Hilliard (CH) Equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^2 u}{\partial x^2}+u^3-u=0, \tag{11}$$

with initial condition $u(x,0)=\dfrac{1}{1+e^{\frac{x}{\sqrt{2}}}}$.

Applying the transformation [23], we get the following partial differential equation

$$\frac{\partial u}{\partial T}-\frac{\partial^2 u}{\partial x^2}+u^3-u=0, \tag{12}$$

The correction functional can be written in the form

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^T \lambda(s)\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+u^3-u\right)ds. \tag{13}$$

The stationary conditions yields

$$1+\lambda=0,\ \lambda'=0.$$

This in turn gives

$$\lambda=-1. \tag{14}$$

Hence (13) takes the form:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^T\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+u^3-u\right)ds. \tag{15}$$

Consequently,

$$u_0(x,T)=\frac{1}{1+e^{\frac{x}{\sqrt{2}}}},\ u_1(x,T)=\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}+\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^2}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}T-\frac{1}{2}\frac{e^{\frac{x}{\sqrt{2}}}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}T-\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}T+\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}T,$$

$$u_2(x,T)=\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}+\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^2}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}T-\frac{1}{2}\frac{e^{\frac{x}{\sqrt{2}}}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}T-\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}T+\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}T$$

$$-\frac{1}{2}\left(\begin{array}{l}-\frac{11}{2}\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^2}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}+9\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^3}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4}+\frac{5}{4}\frac{e^{\frac{x}{\sqrt{2}}}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}+6\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^2}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5}-\frac{3}{2}\frac{e^{\frac{x}{\sqrt{2}}}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4}-6\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^4}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5}\\ \frac{3}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}\left(\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^2}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}-\frac{1}{2}\frac{e^{\frac{x}{\sqrt{2}}}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}-\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}+\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}\right)+\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}-\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}\end{array}\right)T^2,$$

$\vdots$.

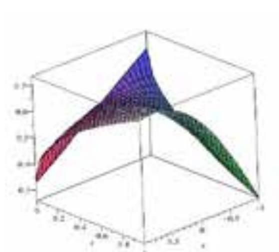
The inverse transformation will yield

$$u_2(x,t)=\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}+\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^2}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{1}{2}\frac{e^{\frac{x}{\sqrt{2}}}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}\frac{t^\alpha}{\Gamma(\alpha+1)}$$

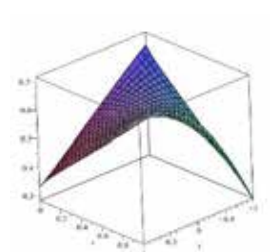
$$-\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$
\begin{aligned}
u_2(x,t) &= \frac{1}{1+e^{\frac{x}{\sqrt{2}}}} + \frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^2}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{1}{2}\frac{e^{\frac{x}{\sqrt{2}}}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\
&- \frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\
&- \frac{1}{2}\left(\begin{array}{l}
-\frac{11}{2}\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^2}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3} + 9\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^3}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4} + \frac{5}{4}\frac{e^{\frac{x}{\sqrt{2}}}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2} + 6\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^2}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5} \\
-\frac{3}{2}\frac{e^{\frac{x}{\sqrt{2}}}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4} - 6\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^4}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5}\frac{3}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}\left(\begin{array}{l}
\frac{\left(e^{\frac{x}{\sqrt{2}}}\right)^2}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3} - \frac{1}{2}\frac{e^{\frac{x}{\sqrt{2}}}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2} \\
-\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3} + \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}
\end{array}\right) \\
+\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3} - \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}
\end{array}\right)\frac{t^{2\alpha}}{\Gamma^2(\alpha+1)}.
\end{aligned}
$$

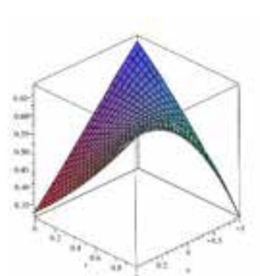
Graphical representation of approximate and exact solutions of (11) for different values of α, using only three iterations of the VIM solution



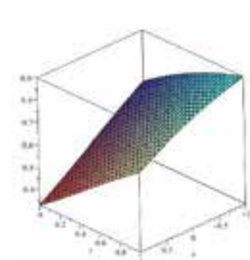
$(a)\,\alpha=0.4$



$(b)\,\alpha=0.8$



$(c)\,\alpha=1$



(d) *Exact solution*

4.3 Gardner Equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^3 u}{\partial x^3} - 6u^2\left(\frac{\partial u}{\partial x}\right) - 6u = 0, \tag{16}$$

with initial condition

$$u(x,0) = -\frac{1}{2}\left(1 - \tanh\left(\frac{x}{2}\right)\right).$$

Applying the transformation [23], we get the following partial differential equation

$$\frac{\partial u}{\partial T} - \frac{\partial^3 u}{\partial x^3} - 6u^2\left(\frac{\partial u}{\partial x}\right) - 6u = 0, \tag{17}$$

The correction functional can be written in the form

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^T \lambda(s)\left(\frac{\partial u}{\partial T} - \frac{\partial^3 u}{\partial x^3} - 6u^2\left(\frac{\partial u}{\partial x}\right) - 6u\right) d\,s. \tag{18}$$

The stationary conditions yields

$$1 + \lambda = 0, \qquad \lambda' = 0.$$

This in turn gives

$$\lambda = -1, \tag{19}$$

Hence (18) takes the form

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^T \left(\frac{\partial u}{\partial T} - \frac{\partial^3 u}{\partial x^3} - 6u^2\left(\frac{\partial u}{\partial x}\right) - 6u\right)(d\,s). \tag{20}$$

Consequently,

$$u_0(x,T) = -\frac{1}{2}\left(1 - \tanh\left(\frac{x}{2}\right)\right),$$

$$u_1(x,T) = -\frac{1}{2}\left(1 - \tanh\left(\frac{x}{2}\right)\right) - 3T - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\tanh^2\left(\frac{x}{2}\right)\right)^2 T + \frac{1}{2}\tanh^2\left(\frac{x}{2}\right)\left(\frac{1}{2} - \frac{1}{2}\tanh^2\left(\frac{x}{2}\right)\right)T$$

$$+3\tanh\left(\frac{x}{2}\right)T + 6\left(-\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right)\right)^2\left(\frac{1}{4} - \frac{1}{4}\tanh^2\left(\frac{x}{2}\right)\right)^2 T,$$

$$u_2(x,T) = -\frac{9}{4}T^3\tanh^9\left(\frac{x}{2}\right) + \frac{63}{16}T^3\tanh^8\left(\frac{x}{2}\right) + \frac{1}{16}\left(90T^2 + 45\,T^3\right).$$

$$\tanh^7\left(\frac{x}{2}\right) + \frac{21}{8}T^3\tanh^{16}\left(\frac{x}{2}\right) + \frac{1}{16}\left(-270T^2 - 275\,T^3\right)\tanh^5\left(\frac{x}{2}\right)$$

$$+\frac{1}{16}\left(-6T + 60\,T^3\right)\tanh^4\left(\frac{x}{2}\right) + \frac{1}{16}\left(-12T + 211T^3 + 200T^2\right).$$

$$\tanh^3\left(\frac{x}{2}\right) + \frac{1}{16}\left(48T^2 + 20T + 54\,T^3\right)\tanh^2\left(\frac{x}{2}\right) +$$

$$\frac{1}{16}\left(124T^2 + 60T + 8 + 37\,T^3\right)\tanh\left(\frac{x}{2}\right) - \frac{31}{8}T - 12T^2 - \frac{1}{2} - \frac{99}{16}T^3$$

The inverse transformation will yield

$$u_1(x,t) = -\frac{9}{4}\tanh^9\left(\frac{x}{2}\right)\frac{t^{3\alpha}}{\Gamma^3(\alpha+1)} + \frac{63}{16}\tanh^8\left(\frac{x}{2}\right)\frac{t^{3\alpha}}{\Gamma^3(\alpha+1)}$$

$$+\frac{1}{16}\left(90\frac{t^{2\alpha}}{\Gamma^2(\alpha+1)} + 45\frac{t^{3\alpha}}{\Gamma^3(\alpha+1)}\right)\tanh^7\left(\frac{x}{2}\right) + \frac{21}{8}\tanh^{16}\left(\frac{x}{2}\right).$$

$$\frac{t^{3\alpha}}{\Gamma^3(\alpha+1)} + \frac{1}{16}\left(-270\frac{t^{2\alpha}}{\Gamma^2(\alpha+1)} - 275\frac{t^{3\alpha}}{\Gamma^3(\alpha+1)}\right)\tanh^5\left(\frac{x}{2}\right) +$$

$$\frac{1}{16}\left(-6\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 60\frac{t^{3\alpha}}{\Gamma^3(\alpha+1)}\right)\tanh^4\left(\frac{x}{2}\right) +$$

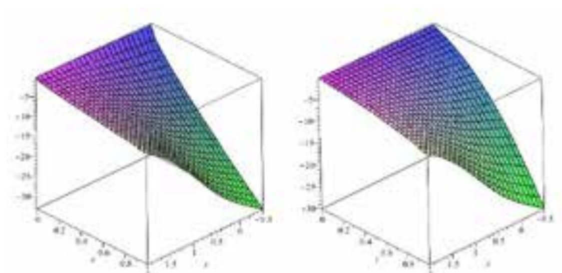
$$\frac{1}{16}\left(-12\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 211\frac{t^{3\alpha}}{\Gamma^3(\alpha+1)} + 200\frac{t^{2\alpha}}{\Gamma^2(\alpha+1)}\right)\tanh^3\left(\frac{x}{2}\right)$$

$$+\frac{1}{16}\left(48\frac{t^{2\alpha}}{\Gamma^2(\alpha+1)} + 20\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 54\frac{t^{3\alpha}}{\Gamma^3(\alpha+1)}\right)\tanh^2\left(\frac{x}{2}\right)$$

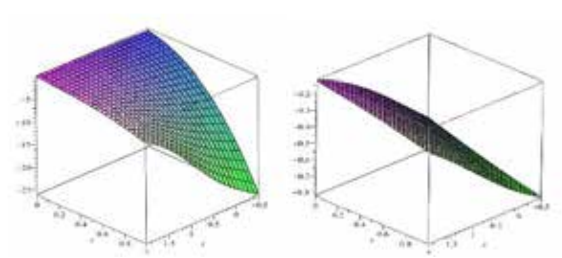
$$+\frac{1}{16}\left(124\frac{t^{2\alpha}}{\Gamma^2(\alpha+1)} + 60\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 8 + 37\frac{t^{3\alpha}}{\Gamma^3(\alpha+1)}\right)\tanh\left(\frac{x}{2}\right)$$

$$-\frac{31}{8}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 12T\frac{t^{2\alpha}}{\Gamma^2(\alpha+1)} - \frac{1}{2} - \frac{99}{16}\frac{t^{3\alpha}}{\Gamma^3(\alpha+1)}.$$

Graphical representation of approximate and exact solutions of (16) for different values of α, using only three iterations of the VIM solution



$(a)\,\alpha = 0.4$ $(b)\,\alpha = 0.8$



$(c)\,\alpha = 0.4$ (d) *Exact solution*

5. CONCLUSIONS

In this paper, variational iteration method (VIM) has been successfully implemented to find approximate solutions for nonlinear partial differential equations of fractional order by considering a change of variable to a new variable.

Three different physical models were tested and the results were in excellent agreement with the exact solution by considering third approximation terms of the variational iteration method. The method is extremely simple, easy to use and is very accurate for solving nonlinear differential difference equation. Also, the method is a powerful tool to search for solutions of various linear/nonlinear problems of fractional order in science and engineering.

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